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Theory of 'self-similarity' of periodic approximants to a quasilattice: III. The case of a non-Bravais-type quasilattice

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Abstract. The mother lattice L of a non-Bravais-type quasilattice (NBTQL) is a non-Bravais-type periodic lattice with higher dimensionality and has an associated Bravais lattice L_0. The main problems on periodic approximants (PAs) to the NBTQL are reduced to similar problems in the case of the relevant Bravais-type quasilattice derived from L_0 and the previous theories on the space groups and 'self-similarity' of the PAs apply to the NBTQL. The present theory includes a general prescription of obtaining PAs to a NBTQL. We apply the theory to several NBTQLs with an octagonal, decagonal or dodecagonal point symmetry. The second important Bravais lattice L_h called the host lattice is associated with L and we have clarified the difference in roles between L_0 and L_h in the theory of PAs to the NBTQL.

1. Introduction

Periodic approximants (PAs) to a quasilattice (QL) are of current interest in connection with approximant crystals to a quasicrystal (Spaepen *et al* 1990, Zhang and Kuo 1990, Edagawa *et al* 1991). We have investigated the space groups and 'self-similarity' of PAs to the octagonal, decagonal and dodecagonal QLs in two dimensions (2D) and the icosahedral one in 3D (Niizeki 1991b, c, 1992b).

A QL in d-dimensions with $d = 2$ or 3 is obtained by the cut-and-projection method from a periodic lattice L in $2d$-dimensions (see, for example, Janssen 1988). L is a Bravais lattice or not, dependent on whether the number of lattice points in a unit cell of L is one or more, respectively. Correspondingly, the QL is classified into Bravais-type or non-Bravais-type (NB-type) (Niizeki 1989a). Only the case of Bravais-type QLs has been considered in our investigations on PAs (Niizeki 1991b, c, 1992a, b). However, there exist several important QLs of NB-type (Niizeki 1988, 1989a, b). The most important of them is the decagonal QL associated with the Penrose tiling with rhombic tiles (de Bruijn 1981). The second important one is a dodecagonal QL associated with a quasiperiodic tiling with squares, regular hexagons and 30°-rhombi (Niizeki 1988, Socolar 1989). More importantly, the QL associated with a real quasicrystal is usually of NB-type.

In this paper, we shall extend our previous theories of PAs to the case of NB-type QLs. We shall confine our considerations to the 2D QLs with the octagonal, decagonal or dodecagonal point symmetry because the present theory is easily extended to the case of the 3D icosahedral QLs. Our theory will include a general prescription for constructing PAs to an NB-type QL.

The translational symmetry of the mother lattice L of an NB-type QL is represented by its Bravais lattice L_0, and a Bravais-type QL obtained from L_0 is naturally associated with the QL. Another important Bravais lattice associated with L is the host lattice, L_h, which is defined as a minimal Bravais lattice among those which include L as their sublattices. We will clarify the difference in roles between the L_0 and L_h in the theory of PAs to the NB-type QL.

We investigate in section 2 general properties of the mother lattices of NB-type n-gonal QLs with $n=8$, 10 or 12 and in section 3 those of the QLs themselves. The contents of these sections are summaries of the papers of the present author (Niizeki 1989a, c) and we will omit, in these sections, references to these papers. We extend, in section 4, the previous theories (Niizeki 1991b, c) of PAs to the case of a NB-type QL and show that the main problems on PAs to the QL are reduced to similar problems in the case of a Bravais-type QL associated with the QL. The results of Niizeki (1991b, c) are used in this section without referring to these papers. We apply in section 5 the theory of the present paper to a representative NB-type n-gonal QL for each case $n=8$, 10 or 12. The contents of sections 2–4 will be easier to understand if they are read in parallel to this section. In section 6 we summarize the results of this paper and discuss several related subjects.

2. The mother lattice of an NB-type quasilattice

We assume that the point symmetry of the NB-type QL is $8mm$, $10mm$ or $12mm$. The mother lattice L of the QL is an NB-type lattice in 4D and embedded in E_4, the 4D Euclidean space. The point group G of L is equal (exactly, isomorphic) to nmm with $n=8$, 10 or 12 and its order, $|G|$, is equal to $2n$. G is generated by r, a 4D rotation with order n, and s, a 4D mirror. r satisfies $r^n=1$ or, more precisely, $P_n(r)=0$ with $P_n(x)$ being the n-cyclotomic polynomial, which is given by $1+x^4$, $1-x+x^2-x^3+x^4$ or $1-x^2+x^4$ for $n=8$, 10 or 12, respectively. Note that $r^{n/2}=-1$, which is nothing but the 4D inversion.

We assume that the space group of L is $pnmm$ with $n=8$, 10 or 12, i.e. symmorphic. Then, L has special points (SPs) with the full point symmetry (nmm) (a point in E_4 is called an SP of L if its point group with respect to L is a centring point group). There exists only a single Bravais class of the 4D n-gonal lattice for each n (Janssen 1988). Let L_0 be the Bravais lattice representing the translational symmetry of L. Then the space group of L is given by $g\equiv G * L_0$, the semi-direct product of G and L_0, provided that the origin of the Cartesian coordinate system for E_4 is chosen appropriately; the lattice points of L_0 are full symmetry points of L. The space goup of L_0 is identical to that of L, so that the SPs are common between L and L_0. The SPs of L_0 are rational points with respect to L_0. As will be shown later, all the lattice points of L are SPs of L_0 for important NB-type QLs.

Let ν be the number of the lattice points of L in a unit cell of L_0. Then L is divided into ν sublattices

$$L=L_1UL_2U\dots UL_\nu \tag{1}$$

with

$$L_i=x_i+L_0 \tag{2}$$

where x_i is a representative of the lattice vectors in L_i and determined in modulo L_0.

We shall confine our arguments to the case where $\boldsymbol{x}_i$ are all rational points with respect to L_0. Then $\boldsymbol{x}_i$ together with L_0 generate a Bravais lattice L_h, which we shall call the host lattice of L. L_h is, in fact, a minimal Bravais lattice among those which include both L and L_0.

L_0 is a superlattice of L_h. The multiplicity m of L_0 with respect to L_h is defined to be the number of the lattice points of L_h in a unit cell of L_0: $m=|L_h/L_0|$, the order of the factor group L_h/L_0. Note that $m \geqslant \nu$. The space group g_h $(=G * L_h)$ of L_h is isomorphic to g $(=pnmm)$ but $g_h \supsetneqq g$.

An n-gonal lattice has two types, Δ and Σ, of mirrors (Niizeki 1991a). There can be two cases with respect to the relative orientation between L_h and L_0. In the normal case, the mirrors of G are of common types between the two lattices but in the inverted case, a type Δ (or Σ) mirror of L_h is of type Σ (or Δ) as a mirror of L_0.

Let $\Lambda \equiv \{1, 2, \ldots, \nu\}$ be the set of the suffices of $\boldsymbol{x}_i$. Then, for given $\sigma \in G$ and $i \in \Lambda$ there exists $j \in \Lambda$ such that $\sigma\boldsymbol{x}_i \equiv \boldsymbol{x}_j \bmod L_0$ or, equivalently, $\sigma L_i = L_j$. This gives rise to a permutational representation of G and we can assume that G acts on Λ as $\sigma i = j$. The symmetry of $\boldsymbol{x}_i$ (or any lattice point of L_i) with respect to L_0 is given by the isotropy group of i: $H_i = \{\sigma | \sigma \in G, \sigma i = i\}$. If $\sigma i \neq i$, then H_i and H_j with $j = \sigma i$ are different but conjugate in G, so that $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ are equivalent but differ only in their 'orientations'. Then, L_i and L_j are equivalent in L. That is, equivalent sublattices in (1) are permuted by the action of an element of G. The number of equivalent sublattices with point group H is equal to $|G|/|H|$. L is called homopolar if all the ν sublattices of L are equivalent but heteropolar otherwise. If L is homopolar, ν represents the number of different 'orientations' of the lattice points of L. On the other hand, if L is heteropolar, it can be divided into several homopolar components, each of which represents a set of equivalent lattice points.

E_4 is decomposed by G into two invariant 2D subspaces E_2 and E_2'. The two subspaces have irrational orientations with respect to L, L_0 and L_h. The point group G acts not only onto E_4 but also onto E_2 and E_2'. We shall call E_2 the physical space and E_2' the internal one.

Let $\boldsymbol{\varepsilon}_i$, $i=1$–4, be the basis vectors of L_h and P (or P') the projectors onto E_2 (or E_2'). Then $\boldsymbol{e}_i \equiv P\boldsymbol{\varepsilon}_i$ (or $\boldsymbol{e}_i' \equiv P'\boldsymbol{\varepsilon}_i$), $i=1$–4, are linearly independent over $\boldsymbol{Z}$. We can assume that $\boldsymbol{e}_i$ (or $\boldsymbol{e}_i'$) are four of the vertex vectors of a regular n-gon centred on the origin of E_2 (or E_2'). We shall call it the unit n-gon. $a \equiv |\boldsymbol{e}_i|$ is called the lattice constant of L_h. A mirror of L_h is of type Δ or Σ dependent on whether it passes a vertex of the unit n-gon or the middle point of its edge.

The set of points, $PL_h \equiv \{\Sigma_i n_i \boldsymbol{e}_i | n_i \in \boldsymbol{Z}\}$, is dense in E_2 and is called a pre-quasilattice. The same is true for $P'L_h$. If a rational point of E_4 with respect to L_h is projected onto E_2 (or E_2'), the resulting vector is written as a linear combination of $\boldsymbol{e}_i$ (or $\boldsymbol{e}_i'$) with rational coefficients and is called a rational point with respect to PL_h (or $P'L_h$).

Let $\boldsymbol{\delta}_i$, $i=1$–4, be the basis vectors of L_0. Then, they are related to $\boldsymbol{\varepsilon}_i$ by a non-singular integer matrix M

$$(\boldsymbol{\delta}_1\boldsymbol{\delta}_2\boldsymbol{\delta}_3\boldsymbol{\delta}_4) = (\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2\boldsymbol{\varepsilon}_3\boldsymbol{\varepsilon}_4)M. \tag{3}$$

Note that $m = |\det(M)|$. M^{-1} is not an integer matrix and we denote by f the smallest positive number such that fM^{-1} is an integer matrix. $f(>1)$ is a divisor of m.

The two basis sets $\{\boldsymbol{\varepsilon}_i\}$ and $\{\boldsymbol{\delta}_i\}$ give rise to two indexing schemes for a 4D vector in E_4; the two indices are related by M. The indexing scheme with $\boldsymbol{\delta}_i$ or $\boldsymbol{\varepsilon}_i$ will be referred to as the canonical scheme or the h-scheme, respectively. All the lattice vectors of L_h are indexed by integers in the h-scheme but this is not the case in the canonical scheme; the indices of ε_i in the canonical scheme are given by the ith column of M^{-1}. This is the reason why the h-scheme is used frequently in the argument on an NB-type QL.

We can consider M to be a matrix representing a linear transformation β satisfying $\delta_i = \beta\varepsilon_i$, $i = 1$–4. It follows that $L_0 = \beta L_h$. It can be shown generally that $\beta = b_0 + b_1 r + b_2 r^2 + b_3 r^3$ with b_i being integers or, equivalently, $M = b_0 I + b_1 R + b_2 R^2 + b_3 R^3$, where R is a unimodular matrix defined by the equation

$$r(\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4) = (\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4)R. \tag{4}$$

Note that R satisfies $R^n = I$ and $P_n(R) = 0$. β decomposes as $\beta = S \oplus S'$, where S and S' are similarity transformations acting onto E_2 and E_2', respectively. We may say that L_0 and L_h are similar because the scale of E_2' is indifferent to the projection method.

In the normal case, $P\delta_i$ is parallel to e_i $(= P\varepsilon_i)$; $P\delta_i = \lambda e_i$, where $\lambda \in Z[2\cos(2\pi/n)]$ is a quadratic algebraic integer. We may write $\beta = \lambda I \oplus \lambda' I$ with λ' being the algebraic conjugate of λ. Note that $m = (\lambda\lambda')^2$ because $\det(M) = \det(\beta)$.

L_h and L_0 have a common special automorphism α which is written as $\alpha = \alpha_0 + a_1 r + a_2 r^2 + a_3 r^3$ with a_i being integers. α acts as a similarity transformation onto E_2 and E_2'. We can assume that α expands E_2 and shrinks E_2'. In the case of $n = 8$ (or 10), α takes a special form $\alpha = \tau I \oplus \tau' I$, where $\tau = 1 + \sqrt{2}$ (or $\tau = (1+\sqrt{5})/2$) and τ' $(= -1/\tau)$ being the algebraic conjugate of τ. τ is an irrational number characterizing the irrational orientation of E_2. The integer matrix N representing α is uni-modular and written as $N = a_0 I + a_1 R + a_2 r^2 + a_3 R^3$.

α is not necessarily an automorphism of L but there exists a finite integer k such that α^k is an automorphism of L. We shall develop our theory by assuming that α is an automorphism of L; α is redefined, if necessary, to be α^k. Then α permutes the sublattices L_i of L and α acts on Λ as a permutation. If L is heteropolar, α may permute its homopolar components.

3. Properties of an NB-type n-gonal QL

An NB-type QL is obtained by the projection method from L as

$$Q(x, \{W(i)\}) = U_i\{P(l+x) \mid l \in L_i, P'(l+x) \in W(i)\} \tag{6}$$

where x is a 4D phase vector and $W(i)$ $(\subset E_2')$ the window assigned to the sublattice L_i. $W(i)$ are usually polygonal domains and their vertices are rational points in E_2' with respect to $P'L_h$, which we shall assume hereafter. The QL is divided naturally into ν sublattices. It is homopolar or heteropolar depending on the nature of L. If heteropolar, it is composed of several homopolar QLs.

The windows must satisfy $\sigma W(i) = W(\sigma i)$ for all $\sigma \in G$. Then, $W(i)$ have the point symmetry H_i. In particular, $W(i)$ has the inversion symmetry, $W(i) = -W(i)$, if H_i is centrosymmetric. On the other hand, if L_i and L_j are equivalent, then $j = \sigma i$ for $\sigma \in G$, so that $\sigma W(i) = W(j)$; $W(i)$ and $W(j)$ are congruent but different in their orientations. If H_i is non-centosymmetric, there exists $j \in \Lambda$ such that $L_i = -L_j$ (or, equivalently, $x_i \equiv -x_j \bmod L_0$) and we obtain $W(j) = -W(i)$.

The macroscopic point symmetry of $Q(x) \equiv Q(x, \{W(i)\})$ is equal to G owing to our choice of the windows. QLs with common windows but different phase vectors form a single local isomorphism class.

A Bravais-type n-gonal QL, $Q_0(x) \equiv Q_0(x, W_0)$, is obtained by the projection method from L_0, where W_0 is a window with point symmetry G. If L is a homopolar lattice formed of a class of SPs of L_0, then a lattice point of $Q(x)$ is a local centre of symmetry of $Q_0(x)$; the point group of the local symmetry is equal to that of the class of SPs. Therefore, $Q(x)$ is formed of 'special points' of $Q_0(x)$. $Q(x)$ is divided into ν sublattices corresponding to ν different orientations of the local symmetry. If $Q(x)$ is heteropolar, a similar argument applies to each of its homopolar component. It is essential in the present argument that the phase vector x is common between $Q(x)$ and $Q_0(x)$.

Since α shrinks E_2', $\alpha W(i)$ is smaller than $W(i)$. We assume that $\alpha W(i) \subset W(\alpha i)$ for all $i \in \Lambda$; if this is not satisfied, we must replace α by some power of α. Let $\bar{W}(i) \equiv \alpha^{-1} W(\alpha i)$ and $\bar{Q}(x) \equiv Q(x, \{\bar{W}(i)\})$. Then $Q(x) \subsetneq \bar{Q}(x)$ because $W(i) \subsetneq \bar{W}(i)$. We can prove by a similar argument as in Niizeki (1991c) that $\bar{Q}(x) = \alpha^{-1} Q(\alpha x)$, which is similar to $Q(\alpha x)$. Therefore, $Q(x)$ is self-similar; $\bar{Q}(x)$ is a deflation of $Q(x)$ and $Q(\alpha x)$ $(= \alpha \bar{Q}(x))$ is a deflation-and rescaling of $Q(x)$.

4. Periodic approximants to an NB-type n-gonal QL

2D lattice planes of L_0 are important in the theory of PAs. Let Π be one of them. Then it is indexed in the canonical scheme by a 2×4 integer matrix K. We may assume that K is irreducible (for reducibility or irreducibility of an integer matrix, see Niizeki 1991c). Then the two columns of K index the two basis vectors of the 2D lattice $\Pi \cap L_0$. The index of Π in the h-scheme is given by $K' = MK$. On the other hand, $\bar{\Pi} \equiv \alpha \Pi$ is another 2D lattice plane, which is indexed by NK and its slope with respect to E_2 is smaller than that of Π.

Since L_0 is a superlattice of L_h, Π is also a 2D lattice plane of L_h. The 2D lattice $\Pi \cap L_0$ is a superlattice of $\Pi \cap L_h$ but the two lattices coincide if K' is irreducible.

A PA to a QL whose mother lattice is L is obtained from a deformed lattice $\tilde{L}$ which is obtained from L by introducing a phason strain; a 2D lattice plane Π of L becomes coincident with E_2 by the deformation. $\tilde{L}$ and the PA are characterized by the index K of Π. A good PA is obtained when the angle between Π and E_2 is small. Then the integers in the index K are related to rational approximants to an irrational of the form $\rho = \tau - k$ with k being an integer smaller than $\tau - 1$; ρ coincides with τ in the case $k = 0$. A series of rational approximants to ρ is obtained from a sequence of the Fibonacci numbers and/or their analogues, which are generated by a recursion relation. ρ may have several series of 'best' rational approximants (Niizeki 1992b).

In the case of an n-gonal QL, PA with two mirrors perpendicular to each other is important (Niizeki 1991b, c, 1992b). The relevant lattice plane Π to the PA is characterized by a pair of rational approximants to ρ as $\langle p/q, u/v \rangle$, where p/q (or u/v) is associated with the first (or second) mirror of Π; the first (or second) column of K is written with p and q (or u and v). The unit cell of the PA is rectangular or rhombic for K irreducible or reducible, respectively. We shall designate a PA with space group X as $X\langle p/q, u/v \rangle$, for example, $pgm\langle 8/5, 5/3 \rangle$.

Π is a lattice plane of L_h as well and indexed in the h-scheme by $K' = MK$, which is, however, not necessarily irreducible. A different pair $\langle p'/q', u'/v' \rangle$ is associated with K'; p' and q' (or u' and v') in K' are related by a 2×2 integer matrix to p and q (or u and v) in the normal case but to u and v (or p and q) in the inverted case. The relation in the normal case is rewritten, alternatively, to the form $p'\tau + q' = \lambda(p\tau + q)$

or a similar form with ρ. It may happen that p' and q' (or u' and v') have a non-trivial common divisor. Then p'/q' (or u'/v') is not a simple fraction and K' is reducible.

We may write the deformed lattice as $\tilde{L} = \Phi L$, where Φ is the linear transformation representing the phason strain; $\Phi\Pi = E_2$. $\tilde{L}_0 \equiv \Phi L_0$ is a Bravais lattice of $\tilde{L}$ and $\tilde{L}_h \equiv \Phi L_h$ the host lattice of $\tilde{L}$. E_2 is a 2D lattice plane of both $\tilde{L}_0$ and $\tilde{L}_h$, so that $\tilde{L}_{0,B} \equiv E_2 \cap \tilde{L}_0$ and $\tilde{L}_{h,B} \equiv E_2 \cap \tilde{L}_h$ are 2D Bravais lattices. It is usual that $\tilde{L}_{0,B}$ is a superlattice of $\tilde{L}_{h,B}$; the two 2D lattices coincide only when K' is irreducible. The shadow lattice of $\tilde{L}_0$ (or $\tilde{L}_h$) is defined by $\tilde{L}_{0,s} \equiv P'\tilde{L}_0$ (or $\tilde{L}_{h,s} \equiv P'\tilde{L}_h$), which is a 2D Bravais lattice in E'_2. We can prove that $m = |\tilde{L}_{h,B}/\tilde{L}_{0,B}| \times |\tilde{L}_{h,s}/\tilde{L}_{0,s}|$. Therefore, if $\tilde{L}_{h,B} = \tilde{L}_{0,B}$, for example, then $|\tilde{L}_{h,s}/\tilde{L}_{0,s}| = m$.

The point group G of L is degraded by Φ to its subgroup $\tilde{G}$, which is the point group of $\tilde{L}$, $\tilde{L}_0$ and $\tilde{L}_h$. The space group is common between $\tilde{L}$ and $\tilde{L}_0$ and given by $\tilde{g}_0 = \tilde{G} * \tilde{L}_0$.

$\tilde{L}$ is decomposed into sublattices $\tilde{L}_i = \tilde{x}_i + \tilde{L}_0$ with $\tilde{x}_i = \Phi x_i$. Therefore, a PA to $Q(x)$ is given as

$$\tilde{Q}(\tilde{x}, \{\tilde{W}(i)\}) = U_i\{P(l+\tilde{x}) | l \in \tilde{L}_i,\ P'(l+\tilde{x}) \in \tilde{W}(i)\} \tag{6}$$

where $\tilde{W}(i)$ are appropriate deformations of $W(i)$ and $\tilde{x} = \Phi x$. A PA $\tilde{Q}_0(\tilde{x})$ to $Q_0(x)$ is obtained similarly from $\tilde{L}_0$.

Two PAs $\tilde{Q}(\tilde{x})$ and $\tilde{Q}_0(\tilde{x})$ are related locally to each other, so that the space group is common between them. Therefore the Bravais lattice of $\tilde{Q}(\tilde{x})$ is given by $\tilde{L}_{0,B}$ and the space group of $\tilde{Q}(\tilde{x})$ is determined by the symmetry of $P'\tilde{x}(\in E'_2)$ with respect to $\tilde{L}_{0,s}$ (Niizeki 1991b). Note that the role of $\tilde{L}_{0,B}$ or $\tilde{L}_{0,s}$ in these arguments can be replaced by $\tilde{L}_{0,h}$ or $\tilde{L}_{h,s}$ only when the former coincide with the latter.

In the case of a high-symmetry PA, it may occur as a singular case that lattice points of $\tilde{L}$ project onto the boundaries of the windows. This causes the PA to have local structures which are not allowed in the ideal QL.

There exists a one-to-one correspondence between PAs to $Q(x)$ and those of $Q_0(x)$. Therefore, we can conclude that a classification of the space groups of the PAs to an NB-type n-gonal QL is completely reduced to a similar problem in the case of the relevant Bravais-type QL; the latter problem has been solved in the series of papers by Niizeki (1991b, c, 1992b). Moreover, we can show as in Niizeki (1991c) that the PAs to $Q(x)$ are grouped into different series in such a way that each series is generated from its prototype member by successive applications of the deflation-and-rescaling. The space group is common among the members of a single series. The procedure of obtaining the deflation or inflation of a given PA to the NB-type QL is similar to the one given in Niizeki (1991c, 1992b) for the case of the Bravais-type QL. The procedure is unique only when $P'x$ as well as the vertices of W_i are rational points in E'_2.

5. Several examples

We shall investigate an octagonal QL, a decagonal one and a dodecagonal one, separately. Some of the properties of the NB-type 4D n-gonal lattices in this section have been investigated in Niizeki (1989a, c) and reference should be made to these articles.

5.1. *The case of an octagonal* QL

The 4D space group $p8mm$ has only one class of SPs with point symmetry $4mm$. The

SPs can assume two different orientations and form a homopolar NB-type octagonal lattice L with two equivalent sublattices. We shall consider an octagonal QL derived from L. Then we obtain $\nu=2$, $m=4$, $f=2$, $\beta=r+r^{-1}$ and $\alpha=1+\beta$. L_0 and L_h have a normal relation with $\lambda=\sqrt{2}$. The basis vectors of L_h is so chosen that $r(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(\varepsilon_2,\varepsilon_3,\varepsilon_4,-\varepsilon_1)$ and $s(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(\varepsilon_4,\varepsilon_3,\varepsilon_2,\varepsilon_1)$. The first of the two equations determines R and $M\equiv R+R^{-1}$ is obtained as given in the appendix. Note that $M^2=2I$ and $M^{-1}=M/2$. It follows that $L_0=\{\Sigma_i n_i\varepsilon_i|n_i\in Z,\ n_1\equiv n_3$ and $n_2\equiv n_4 \bmod 2\}$, $x_1=\varepsilon_1$ and $x_2=\varepsilon_2$. Moreover, we obtain $L_h=L_0UL_1UL_2UL_3$ with $L_3\equiv\varepsilon_1+\varepsilon_2+L_0$, which is formed of SPs with full symmetry of L_0. L_i are invariant against α.

L_0 and L_3 form an octagonal black-and-white Bravais lattice (Niizeki 1990b) and $L_{03}\equiv L_0UL_3$ is another octagonal lattice of the Bravais-type. L is a simple translation of this lattice; $L=\varepsilon_1+L_{03}$. Note, however, that L is considered to be of the NB-type because the point symmetry of its lattice points is assumed to be $4mm$ but not $8mm$.

Let $W(1)$ (or $W(2)$) be a square window whose vertices are at $\pm e_2'$ and $\pm e_4'$ (or $\pm e_1'$ and $\pm e_3'$). Then L together with these windows yields an NB-type octagonal QL, $Q(x)$, as shown in figure 1. An inflation (or deflation) of $Q(x)$ is obtained by shrinking (or expanding) the windows $W(i)$ as $\tau^{-1}W(i)$ (or $\tau W(i)$). The inflated QL is superimposed in figure 1. The bond length of the QL is equal to $|e_1+e_2|=2a\cos(\pi/8)$.

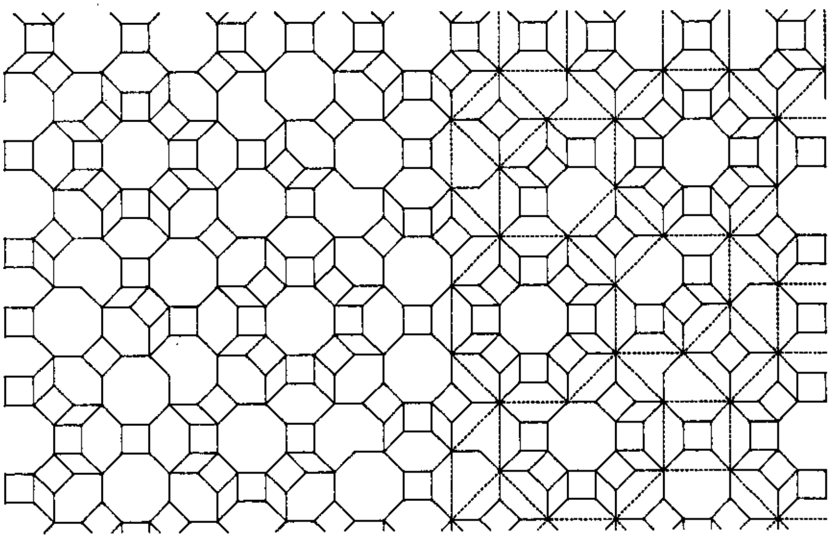

Figure 1. An NB-type octagonal quasilattice (solid lines) and a part of its inflation (dashed lines). The lattice points are given by the positions of the vertices of the octagonal quasiperiodic tiling composed of five kinds of tiles, one of which is a concave octagon. The QL is composed of two sublattices and two vertices connected by a bond belong to different sublattices. Each kind of tile in dashed lines has its own decoration but a rhombic tile and a hexagonal tile have polarities.

The Bravais-type octagonal QL, $Q_0(x, W_0)$, obtained from L_0 is the set of vertices of the Ammann tiling shown in figure 2, where W_0 is chosen to be a regular octagon whose vertices are at $r^i(e_1'+e_4')$, $i=0$–7. $Q(x)$ and $Q_0(x)$ interpenetrate each other and the set of all the centres of square tiles of the Ammann tiling is exactly equal to $Q(x)$.

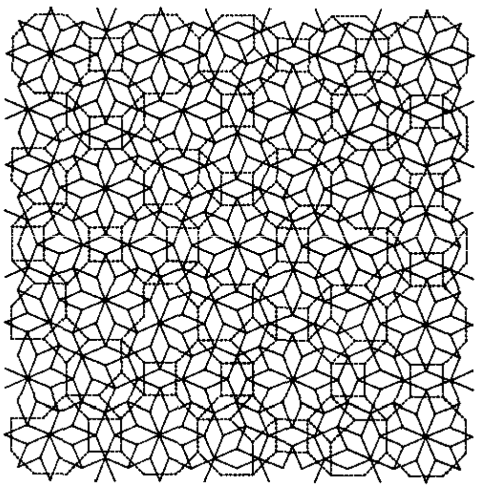

Figure 2. The octagonal Ammann tiling (solid lines) associated with the NB-type octagonal QL (dashed lines). The NB-type QL coincides with the set of all the centres of square tiles of the Ammann tiling. Conversely, the latter tiling is obtained from the former by appropriate decorations of the tiles provided that an appropriate 'polarity' is introduced into each square tile of the NB-type QL.

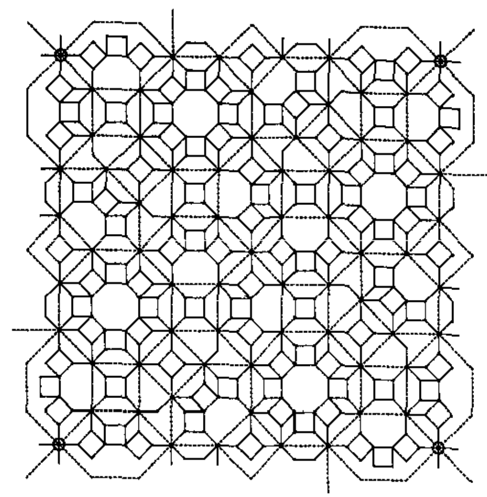

Figure 3. A square PA (solid lines) to an NB-type octagonal QL and its inflation (dashed lines). The original PA is desinad by $p4g\langle 12/5, 12/5\rangle$, and the inflated PA by $p4g\langle 5/2, 5/2\rangle$. The space group is common between the two PAs. The unit cell is a square whose corners are shown by circles. The corners of the cell and its centre are the centres of the four-fold symmetry.

The bond length of $Q_0(x)$ is equal to $\sqrt{2}a$. The double inflation (τ^2-scaling) of $Q_0(x)$ is the third octagonal QL, which is composed of the eight-pronged vertices of the Ammann tiling; the third QL is identical to the set of centres of the octagonal tiles of $Q(x)$.

Let us investigate square PAs to the octagonal QL. The relevant deformed lattice is characterized by a fraction p/q approximating τ $(=1+\sqrt{2})$ and indexed by $K = [qppq/\bar{p}\bar{q}qp]$, where the first (or last) four integers in K show the first (or second) column of K (Niizeki 1991c). p and q must have opposite parities in order that K is irreducible. p' and q' in K' $(=MK)$ are determined by the equation $p'\tau + q' = \sqrt{2}(p\tau + q)$, which yields $p' = p + q$ and $q' = p - q$. It follows that $p' + q'$ is even and K' is reducible. $\tilde{L}_{\mathrm{h,B}}$ is a square lattice which is the centred version of $\tilde{L}_{0,\mathrm{B}}$. The shadow lattice, $\tilde{L}_{0,\mathrm{s}}$, is also a square lattice. If $P'\tilde{x}$ is located on the centre of a square unit cell of $\tilde{L}_{0,\mathrm{s}}$, the space group of $\tilde{Q}(\tilde{x})$ is $p4g$ (Niizeki 1991c).

We show in figure 3 the PA $p4g\langle 12/5, 12/5\rangle$ together with its inflation. The inflated PA is designated by $p4g\langle 5/2, 5/2\rangle$. The original PA and its inflation are related locally in a way similar to that of the relevant QL and its inflation.

We show in figure 4 two PAs, $\tilde{Q}(\tilde{x})$ and $\tilde{Q}_0(\tilde{x})$, which are designated by $p4g\langle 5/2, 5/2\rangle$. $\tilde{Q}(\tilde{x})$ is identical to the set of the centre of all the square tiles in $\tilde{Q}_0(\tilde{x})$. The two PAs are related locally in a way similar to that of the relevant QLs.

5.2. The case of a decagonal QL

The 4D space group $p10mm$ has two classes of SPs with point symmetry $5m$, which is non-centrosymmetric. The SPs form a heteropolar NB-type decagonal lattice with two

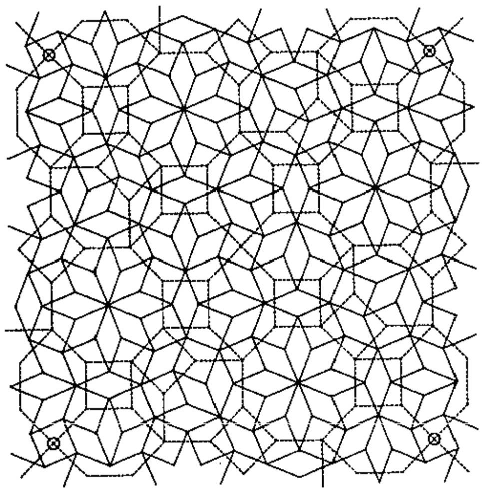

Figure 4. A square PA (dashed lines) to an NB-type octagonal QL and that to a Bravais-type (solid lines). The two PAs are designated by $p4g(5/2, 5/2)$. They are related locally to each other in a similar way to that of their originals (see figure 2). The lattice points of the PA with dashed lines are located on the centres of square tiles of the other PA.

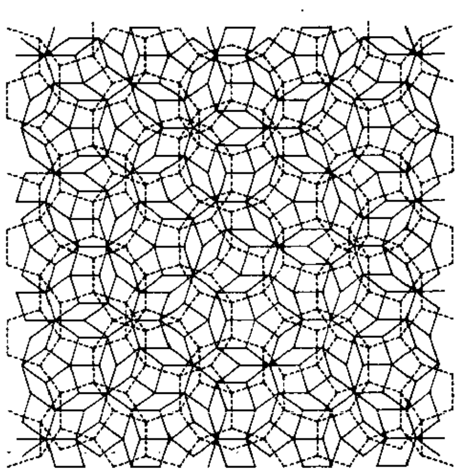

Figure 5. Two decagonal QLs associated with the Penrose tiling with rhombic tiles and the one with pentagonal tiles. The vertices of the rhombic (or pentagonal) Penrose tiling form an NB-type (or Bravais-type) decagonal QL. The NB-type QL is composed of two homopolar components, one of which is the set of centres of the pentagonal tiles in the second Penrose tiling. Conversely, the lattice points of the Bravais-type QL are located on special positions of the fat rhombi of the rhombic Penrose tiling; the special positions are determined by the well known polarities of the fat rhombi.

homopolar components. We will consider this lattice and denote it L. Then we obtain $\nu=4$, $m=5$, $f=5$, $\beta=r-r^{-1}$ and $\alpha=r+r^{-1}$. The orientations are inverted between L_h and L_0. It is convenient to take for L_h (or L_0) a symmetrical but overcomplete set (Niizeki 1990a) of basis vectors ε_i, $i=0$–4, with $\Sigma_i\varepsilon_i=0$ (or δ_i with $\Sigma_i\delta_i=0$): ε_i satisfy $r^2(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)=(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_0)$ and $s(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)=-(\varepsilon_0, \varepsilon_4, \varepsilon_3, \varepsilon_2, \varepsilon_1)$. δ_i is related to ε_i by $\delta_i=\varepsilon_{i+2}-\varepsilon_{i-2}$ with the convention $\varepsilon_{i+5}=\varepsilon_i$. L_0 is written with ε_i as $L_0=\{\Sigma_i n_i\varepsilon_i | n_i\in \boldsymbol{Z}, \Sigma_i n_i=0\}$ and we obtain $x_i=i\varepsilon_1$, $i=1$–4. Note that $x_3\equiv -x_2$ and $x_4\equiv -x_1 \bmod L_0$ and also that $L_h=L_0UL$. The matrix M is given in the appendix. The two homopolar components of L are $L^{(1)}\equiv L_1UL_4$ and $L^{(2)}\equiv L_2UL_3$. r interchanges the two sublattices of $L^{(1)}$ and also those of $L^{(2)}$, while α interchanges $L^{(1)}$ and $L^{(2)}$. More precisely, α permutes (L_1, L_2, L_3, L_4) as (L_3, L_1, L_4, L_2). The permutations r and α satisfy $r=\alpha^2$ and $r^2=\alpha^4=E$ with E being the identity permutation.

The vertices of the rhombic Penrose tiling as shown in figure 5 form an NB-type decagonal QL, $Q(x)$, which is obtained with the projection method from L by assuming appropriate pentagonal windows for $W(i)$ (de Bruijn 1981, Janssen 1988, Niizeki 1989a). We shall call $Q(x)$ a Penrose QL. It is composed of two homopolar components $Q^{(1)}$ and $Q^{(2)}$ which are derived from the two sublattices $L^{(1)}$ and $L^{(2)}$ of L. $Q^{(1)}$ and $Q^{(2)}$ are similar; the former is τ-times the latter (Niizeki 1989a). Let D be a regular decagon whose vertices are at $\pm P'\delta_i$, $i=0$–4. Then the Bravais-type decagonal QL, $Q_0(x, D)$, obtained from L_0 by using D as the window yields the pentagonal Penrose tiling as shown in figure 5. The set of all the centres of the pentagonal tiles in $Q_0(\boldsymbol{x})$ is identical to $Q^{(1)}$ as shown in the same figure. The bond length of $Q(\boldsymbol{x})$ (or $Q_0(\boldsymbol{x})$) is equal to a (or $2a\sin(\pi/5)$).

The deformed lattice associated with PAs with two mirrors is characterized by the index $K=[pq00q/0uv\bar{v}\bar{u}]$ with p/q and u/v being rational approximants to the golden ratio τ $(=(1+\sqrt{5})/2)$; K is written as $[\bar{t}\bar{p}\bar{p}\bar{t}/uv\bar{v}\bar{u}]$ with $t=p-q$ in the asymmetrical index scheme used in Niizeki (1991b). The first or second column of K refers to the Δ or Σ direction of $\tilde{L}_0$, respectively. The relevant integers p', q', u' and v' in K' $(=MK)$ are given by $p'=u+2v$, $q'=2u-v$, $u'=q$ and $v'=p-q$, which satisfy $p'\tau+q'=\sqrt{5}(u\tau+v)$ and $u'\tau+v'=\tau^{-1}(p\tau+q)$.

$\tilde{L}_{0,\mathrm{B}}$ is a rhombic lattice if the conditions, $p\equiv v$ and $t\equiv u \bmod 2$, are satisfies but is a rectangular one otherwise. $\tilde{L}_{\mathrm{h,B}}$ belongs to the same Bravais class as that of $\tilde{L}_{0,\mathrm{B}}$.

Fibonacci numbers, F_k, yield best approximants to τ and the Lucas numbers, L_k $(=F_{k-1}+F_{k+1})$, second best ones. We shall confine our arguments to these two types of approximants. Then the type is common between p/q and u'/v' $(=q/(p-q))$ but it is opposite between u/v and p'/q'. We obtain $\tilde{L}_{0,\mathrm{B}}=\tilde{L}_{\mathrm{h,B}}$ if $u/v=F_{k+1}/F_k$. On the other hand, $\tilde{L}_{0,\mathrm{B}}$ is a superlattice of $\tilde{L}_{\mathrm{h,B}}$ if $u/v=L_{k+1}/L_k$ because we obtain $p'=5F_{k+1}$ and $q'=5F_k$. The unit cell of $\tilde{L}_{0,\mathrm{B}}$ in this case is five times that of $\tilde{L}_{\mathrm{h,B}}$.

We show in figure 6 a rectangular PA to the Penrose QL. The PA is designated by $pgm\langle 8/5,5/3\rangle$.

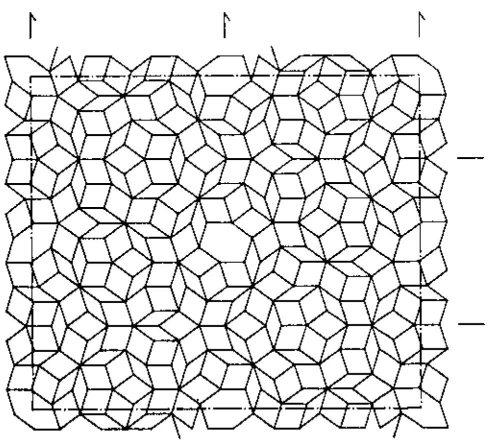

Figure 6. A rectangular PA, $pgm\langle 8/5,5/3\rangle$, to the Penrose QL. The rectangle with dot-dashed lines is the unit cell. The horizontal mirrors and vertical glides are shown by lines and arrows, respectively. There exist two kinds of hexagonal defects because the singular case has happened; the centres of the hexagons are the centres of the inversion symmetry. An acute (or obtuse) hexagon will be divided into two skinny (or fat) rhombi and one fat (or skinny) rhombus if the phase vector is shifted infinitesimally along the horizontal axis or the vertical one in the internal space. However, the vertical glides or the orizontal mirrors will be lost then, respectively.

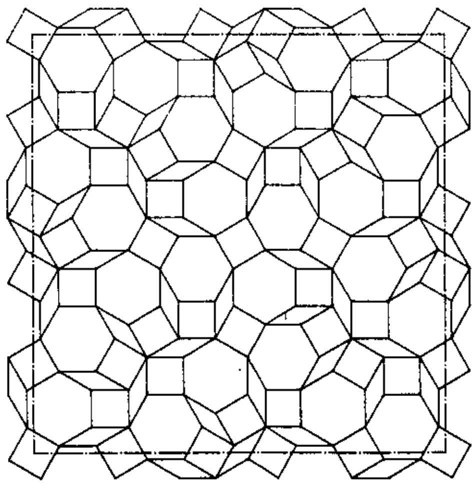

Figure 7. A square PA, $p4g\langle 3/2,3/2\rangle$, to the NB-type dodecagonal QL. The corners and the centre of the square unit cell are the centres of the four-fold symmetry.

5.3. The case of a dodecagonal QL

The 4D space group $p12mm$ has two classes of SPs with point symmetry $3m$, which is non-centrosymmetric. We consider here a homopolar NB-type dodecagonal lattice derived from one of the two classes. Then we obtain $\nu=4$, $m=9$, $f=3$ and $\beta=r+r^{-1}$. L_0 and L_h have a normal relation with $\lambda=\sqrt{3}$. The basis vectors of L_h is so chosen that $r(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)=(\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_3-\varepsilon_1)$ and $s(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)=(\varepsilon_4, \varepsilon_3, \varepsilon_2, \varepsilon_1)$. The matrix M is given in Appendix. Note that $M^2=3I$ and $M^{-1}=M/3$. We may write $L_0=\{\Sigma_i n_i\varepsilon_i | n_i \in Z, n_1 \equiv n_3$ and $n_2 \equiv n_4 \bmod 3\}$, while $x_1=\varepsilon_1+\varepsilon_2, x_2=\varepsilon_2+\varepsilon_3, x_3=-x_1$ and $x_4=-x_2$. The relevant automorphism of L is given by $\alpha=2+\beta=\tau I\oplus\tau^{-1}I$ with $\tau=2+\sqrt{3}$. Note that $\alpha_0 \equiv 1+r$ is an automorphism of L_0 but not of L. r permutes L_i cyclically, while α permutes their order pairwisely into (L_3, L_4, L_1, L_2). r and α satisfy $r^2=\alpha$ and $r^4=\alpha^2=E$.

Another homopolar NB-type dodecagonal lattice obtained from the other class of SPs with point group $3m$ is similar to L because it is written as $\alpha_0 L$.

The deformed lattice with two mirrors of type Δ is characterized by the index $K=[p, 2q, 0, -q/-v, 0, 2v, u]$, where p/q and u/v are rational approximants to $\sqrt{3}$ (Niizeki 1992b). The relevant integers p' and q' in K' $(=MK)$ are given by $p'=3q$ and $q'=p$, which satisfy $p'+\sqrt{3}q'=\sqrt{3}(p+\sqrt{3}q)$. u' and v' are given similarly. If p is not divided by 3, p'/q' is a simple fraction and $\tilde{L}_{0,\mathrm{B}}=\tilde{L}_{\mathrm{h,B}}\cdot\sqrt{3}$ has three series of best approximants and the numerator of every approximant in one of the three is divided by 3 (Niizeki 1992b). Therefore, $\tilde{L}_{0,\mathrm{B}}$ is a superlattice of $\tilde{L}_{\mathrm{h,B}}$ if p/q or u/v belongs to this series.

We show in figure 7 a square PA to the dodecagonal QL. The PA is designated by $p4g\langle 3/2, 3/2\rangle$.

6. Summary and discussions

The arguments made so far are summarized as follows: The mother lattice L of an NB-type QL has two associated Bravais lattices L_0 and L_h. Of the two, L_0 is of essential importance in the symmetry properties of L because it represents the translational part of the space group of L. There exists a one-to-one correspondence between PAs to the NB-type QL and to a Bravais-type QL obtained from L_0, so that the main problems on the former PAs are reduced to similar problems on the latter and the previous theories on the space groups and 'self-similarity' of the PAs apply essentially to the NB-type QL. Our theory includes a general prescription of obtaining PAs to an NB-type QL. These results are confirmed by applying the theory to several NB-type QLs with octagonal, decagonal or dodecagonal point symmetry.

We consider here the reason why $\tilde{L}_{\mathrm{h,B}}$ does not always represent the Bravais lattice of the relevant PA. $\tilde{L}_h$ is composed of m sublattices which are translationally equivalent to $\tilde{L}_0$ and $\tilde{L}_0$ is one of them. Using these we can show easily the following proposition: A necessary and sufficient condition for $\tilde{L}_{\mathrm{h,B}}$ and $\tilde{L}_{0,\mathrm{B}}$ to coincide is that all the lattice points of $\tilde{L}_{\mathrm{h,B}}$ belong to $\tilde{L}_0$.

We have shown that the approximant lattice $\tilde{L}_0$ is characterized by $\langle p/q, u/v\rangle$, i.e. a pair of approximants to τ (or a similar irrational) and $\tilde{L}_h$ by $\langle p'/q', u'/v'\rangle$. A best PA to the NB-type QL is obtained when $\tilde{L}_0$ is a best approximant to L_0 because the Bravais lattice of the PA is determined by $\tilde{L}_0$. Note, however, that $\tilde{L}_h$ is not necessarily then a best approximant to L_h.

We can derive from these arguments the following conclusion: It is not L_h but L_0 that dominates the properties of the PAs to the relevant QL although the h-scheme is used frequently in an argument on an NB-type QL. One must not confuse the two lattices.

The 4D n-gonal lattice has many classes of SPs (Niizeki 1989c) and we can construct other kinds of NB-type QLs than those investigated in section 5. We shall discuss briefly two of them. First, a homopolar NB-type octagonal QL is derived from the Ammann octagonal tiling in figure 2 by putting lattice points onto all the centres of rhombic tiles (45°-rhombi). The mother lattice of this QL is an NB-type 4D octagonal lattice with $\nu=4$, $m=8$, $f=2$, $\beta=1+r+r^2+r^3$ and $\boldsymbol{x}_i=\boldsymbol{\varepsilon}_i$ ($i=1$–4). Second, a homopolar NB-type dodecagonal QL is derived from the dodecagonal tiling in section 5 by putting lattice points onto all the centres of square tiles. This QL is identical to figure 3 in Nissen (1990). The mother lattice of this QL is an NB-type 4D dodecagonal lattice (Niizeki 1989a) with $\nu=3$, $m=4$, $f=2$, $\beta=1+r^3$ and $\boldsymbol{x}_i=\boldsymbol{\varepsilon}_i$ ($i=1$–4). Note that L_0 and L_h have an inverted relation for the two cases presented here.

Minimal dimensionality of the mother lattice of the n-gonal QL with $n=8$, 10 or 12 is four and we have used 4D n-gonal lattices to obtain the NB-type n-gonal QLs. However, the Penrose QL (or the dodecagonal QL in section 5.3) is obtained, alternatively, from the 5D (or 6D) simple hypercubic lattice as shown by de Bruijn (1981) (or by Niizeki 1988 and Socolar 1989), which is a Bravais lattice. An n-gonal QL is obtained even from the simple hypercubic lattice in n-dimensions (Gähler and Rhyner 1986, Whittaker and Whittaker 1988). The PAs to the Penrose QL have been investigated in this framework although the space groups of the PAs have not been fully investigated (Eintin–Wohlman *et al* 1988, Edagawa *et al* 1991). A QL obtained from a mother lattice with non-minimal dimensionality is always of NB-type in the definition of the present paper even though the mother lattice is a Bravais lattice. Therefore the PAs to the QL are treated by the formalism developed in this paper.

We can obtain many NB-type n-gonal QLs by the dual grid method (Niizeki 1989b, Stampfli 1990). They are obtained by the projection method (Niizeki 1989b) as well, so that their PAs are also treated by the formalism developed in this paper.

Acknowledgments

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Appendix

The transformation matrix M for (a) the octagonal case, (b) the decagonal case and (c) the dodecagonal case:

$$(a)\ \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 0 & 1 & 0 & -1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

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